Invariant stochastic orders

Magdalena Frąszczak

Wrocław, 04.01.2012

Preliminaries

- $X \sim F$, $Y \sim G$ random variable with density function f and g respectively;
- $F^{-1}(u) = \inf\{x : F(x) \geqslant u\}$, $u \in (0,1)$ - the quantile function of F;
- $\frac{d}{du}F^{-1}(u) = 1/fF^{-1}(u)$ – the *quantile-density* of the distribution F
- $fF^{-1}(u)$ – the *density-quantile* function.
  (for G analogously)
- $\mathcal{F}$ – the class of absolutely continuous distribution functions F with $F(0) = 0$;
- $\mathbf{\Phi}$ – the class of continuous positive functions φ defined on $(0,1)$.

Stochastic orders

- the likelihood ratio order
  $F \leqslant_{\mathrm{lr}} G \;\equiv\; g(x)/f(x)$ is increasing;
- the hazard rate order
  $F \leqslant_{\mathrm{hr}} G \equiv \overline{G}(x)/\overline{F}(x)$ is increasing;
- the reversed hazard rate order
  $F \leqslant_{\mathrm{rh}} G \equiv G(x)/F(x)$ is increasing;
- the usual stochastic order
  $F \leqslant_{\mathrm{st}} G \equiv F(x) \geqslant G(x)$ for all x;

Stochastic orders

- the dispersive order
  $F \leqslant_{\mathrm{disp}} G \equiv G^{-1}F(x) - x$ is increasing;
- the convex order
  $F \leqslant_{\mathrm{c}} G \equiv G^{-1}F$ is convex on S_F;
- the star order
  $F \leqslant_{*} G \equiv G^{-1}F$ is star-shaped on S_F;
- q – order
  $F \leqslant_{\mathrm{q}} G \equiv \limsup_{u \to 1}[F^{-1}(u)/G^{-1}(u)] < \infty.$

The generalized TTT transform

- The total time on test transform (TTT transform)

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx$$

The generalized TTT transform

- The total time on test transform (TTT transform)

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx$$

- The generalized total time on test transform (GTTT transform) of the distribution F with respect to function φ

$$H_F^{-1}(t,\varphi) = \int_{-\infty}^{F^{-1}(t)} \varphi F(x)dx$$

The generalized TTT transform

- The total time on test transform (TTT transform)

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx$$

- The generalized total time on test transform (GTTT transform) of the distribution F with respect to function φ

$$H_F^{-1}(t, \varphi) = \int_{-\infty}^{F^{-1}(t)} \varphi F(x)dx$$

- GTTT transform of the distribution F with respect to distribution $G \in \mathcal{F}$

$$H^{-1}(t; F|G) = \int_0^{F^{-1}(t)} gG^{-1}F(x)dx$$

Lemma 1

Let F and G be absolutely continuous distributions with densities f and g respectively and $F(0) = G(0) = 0$. Then:

- $F \leqslant_{\text{disp}} G \Leftrightarrow fF^{-1}(u)/gG^{-1}(u) \geqslant 1, \quad u \in (0,1)$.
- $F \leqslant_{\text{c}} G \Leftrightarrow fF^{-1}(u)/gG^{-1}(u)$ is increasing in $u \in (0,1)$.

Lemma 1

Let F and G be absolutely continuous distributions with densities f and g respectively and $F(0) = G(0) = 0$. Then:

- $F \leqslant_{\mathrm{disp}} G \Leftrightarrow fF^{-1}(u)/gG^{-1}(u) \geqslant 1, \quad u \in (0,1)$.
- $F \leqslant_{\mathrm{c}} G \Leftrightarrow fF^{-1}(u)/gG^{-1}(u)$ is increasing in $u \in (0,1)$.

Lemma 2

Let F and G be absolutely continuous distributions with densities f and g respectively and such that $\lim_{u\to 1} G^{-1}(u) = \infty$ and $\lim_{u\to 1}[gG^{-1}(u)/fF^{-1}(u)]$ exists. Then

$$F \leqslant_{\mathrm{q}} G \quad \Leftrightarrow \quad \lim_{u\to 1}[gG^{-1}(u)/fF^{-1}(u)] < \infty.$$

Theorem 1

Let $F, G \in \mathcal{F}$ and $\varphi_1, \varphi_2 \in \mathbf{\Phi}$. Then

- if $F \leqslant_{\mathrm{c}} G$ and $\varphi_2(u)/\varphi_1(u)$ is increasing in $u \in (0,1)$, then $H_F(\cdot;\varphi_1) \leqslant_{\mathrm{c}} H_F(\cdot;\varphi_2) \leqslant_{\mathrm{c}} H_G(\cdot;\varphi_2)$;
- if $F \leqslant_{\mathrm{disp}} G$ and $\varphi_2(u)/\varphi_1(u) \geqslant 1$ for $u \in (0,1)$, then $H_F(\cdot;\varphi_1) \leqslant_{\mathrm{disp}} H_F(\cdot;\varphi_2) \leqslant_{\mathrm{disp}} H_G(\cdot;\varphi_2)$.
- if $F \leqslant_{*} G$ and φ is decreasing, then $H_F(\cdot;\varphi) \leqslant_{*} H_G(\cdot;\varphi)$.

Theorem 1

Let $F, G \in \mathcal{F}$ and $\varphi_1, \varphi_2 \in \mathbf{\Phi}$. Then

- if $F \leqslant_{\mathrm{c}} G$ and $\varphi_2(u)/\varphi_1(u)$ is increasing in $u \in (0,1)$, then $H_F(\cdot;\varphi_1) \leqslant_{\mathrm{c}} H_F(\cdot;\varphi_2) \leqslant_{\mathrm{c}} H_G(\cdot;\varphi_2)$;
- if $F \leqslant_{\mathrm{disp}} G$ and $\varphi_2(u)/\varphi_1(u) \geqslant 1$ for $u \in (0,1)$, then $H_F(\cdot;\varphi_1) \leqslant_{\mathrm{disp}} H_F(\cdot;\varphi_2) \leqslant_{\mathrm{disp}} H_G(\cdot;\varphi_2)$.
- if $F \leqslant_* G$ and φ is decreasing, then $H_F(\cdot;\varphi) \leqslant_* H_G(\cdot;\varphi)$.

Theorem 2

Let $\varphi_1, \varphi_2 \in \mathbf{\Phi}$ and F and G satisfy the assumptions of Lemma 2. If $F \leqslant_q G$ and $\lim_{u \to 1}[\varphi_2(u)/\varphi_1(u)] < \infty$, then $H_F(\cdot;\varphi_1) \leqslant_q H_F(\cdot;\varphi_2) \leqslant_q H_G(\cdot;\varphi_2)$.

Lehmann and Rojo (1992) obtained stochastic orders which can be defined by properties of function $k(u) = GF^{-1}(u),\ u \in (0,1)$

Lemma (Lehmann, Rojo 1992)

(i) $F \leqslant_{\mathrm{lr}} G \quad \Leftrightarrow \quad k(u)$ is convex;
(ii) $F \leqslant_{\mathrm{hr}} G \quad \Leftrightarrow \quad 1 - k^{-1}(1-u)$ is star-shaped;
(iii) $F \leqslant_{\mathrm{rh}} G \quad \Leftrightarrow \quad k(u)$ is star-shaped;
(iv) $F \leqslant_{\mathrm{st}} G \quad \Leftrightarrow \quad k(u) \leqslant u$ for all $u \in (0,1)$.

This stochastic orders are invariant under monotone transformations

Invariant directional preorderings

Preorder of the distributions $F \in \mathcal{F}$ is a set $\mathcal{S}$ of ordered pairs (F, G) in $\mathcal{F} \times \mathcal{F}$ satisfying:

1. $(F, F) \in \mathcal{S}$ for all $F \in \mathcal{F}$
2. $(F, G) \in \mathcal{S}$ and $(G, H) \in \mathcal{S}$ implies $(F, H) \in \mathcal{S}$.

Invariant directional preorderings

Preorder of the distributions $F \in \mathcal{F}$ is a set $\mathcal{S}$ of ordered pairs (F, G) in $\mathcal{F} \times \mathcal{F}$ satisfying:

1. $(F, F) \in \mathcal{S}$ for all $F \in \mathcal{F}$
2. $(F, G) \in \mathcal{S}$ and $(G, H) \in \mathcal{S}$ implies $(F, H) \in \mathcal{S}$.

Definition

The preorder $\mathcal{S}$ is invariant under the GTTT transform, if

$$F \leqslant_S G \quad \Rightarrow \quad H_F(\cdot;\varphi) \leqslant_S H_G(\cdot;\varphi) \quad \text{for all} \quad \varphi \in \mathbf{\Phi}.$$

Invariant directional preorderings

Preorder of the distributions $F \in \mathcal{F}$ is a set $\mathcal{S}$ of ordered pairs (F, G) in $\mathcal{F} \times \mathcal{F}$ satisfying:

1. $(F, F) \in \mathcal{S} \quad$ for all $\quad F \in \mathcal{F}$
2. $(F, G) \in \mathcal{S} \quad$ and $\quad (G, H) \in \mathcal{S} \quad$ implies $\quad (F, H) \in \mathcal{S}$.

Definition

The preorder $\mathcal{S}$ is invariant under the GTTT transform, if

$$F \leqslant_S G \quad \Rightarrow \quad H_F(\cdot;\varphi) \leqslant_S H_G(\cdot;\varphi) \quad \text{for all} \quad \varphi \in \mathbf{\Phi}.$$

$$\mathbf{O}(F, G) = \{(H_F(\cdot;\varphi), H_G(\cdot;\varphi)), \varphi \in \mathbf{\Phi}\}$$

Theorem 3.

The function

$$\kappa(u) = \frac{fF^{-1}(u)}{gG^{-1}(u)}, \qquad u \in (0,1),$$

is a maximal invariant under the GTTT transforms with respect to the class $\mathbf{\Phi}$.

Ordering the orbits and distances between ordered distributions

Let F_1 and G_1 from $\mathcal{F}$ and define

$$F_2 = H_{F_1}(\cdot;\varphi),\ \ G_2 = H_{G_1}(\cdot;\varphi),\ \ \varphi \in \mathbf{\Phi}.$$

.

Since (F_1, G_1) and (F_2, G_2) are in the same orbit, we say F_1 and G_1 are at the same distance from each other as F_2 and G_2.

If $F_1 \leqslant_S G_1 \leqslant_S G_2$, we say that G_2 is further to the right of F_1 than G_1 is.
This situation implies that $G_1 = H_{F_1}(\cdot;\varphi)$ and $H_{F_1}(\cdot;\varphi) \leqslant_S G_2$ for some $\varphi \in \mathbf{\Phi}$.

For two pairs (F_1, G_1), (F_2, G_2) with $F_i \leqslant_S G_i$, $i = 1, 2$, we say that G_2 is further to the right of F_2 than G_1 is of F_1, if $F_2 = H_{F_1}(\cdot; \varphi)$ and $H_{G_1}(\cdot; \varphi) \leqslant_S G_2$. It is easy to see that in terms of $\kappa_1 = f_1 F_1^{-1} / g_1 G_1^{-1}$ and $\kappa_2 = f_2 F_2^{-1} / g_2 G_2^{-1}$ this is equivalent to

$$\frac{\kappa_2}{\kappa_1} \in \mathbf{K}_S,$$

where $\mathbf{K}_S$ is a class of functions which defines the preorder S.

Let $\kappa_i = f_i F_i^{-1} / g_i G_i^{-1}$, $i = 1, 2$

Definicion

Let $F_1 \leqslant_S G_1$, $F_2 \leqslant_S G_2$. Then G_2 is said to be further to the right of F_2 than G_1 is of F_1 if

$$\frac{\kappa_2}{\kappa_1} \in \mathbf{K}_S.$$

A metric must satisfy the following conditions:

- $d(F, G) = d(H_F(\cdot;\varphi), H_G(\cdot;\varphi)), \quad$ for all $\quad \varphi \in \mathbf{\Phi}$ and
- $d(F_1, G_1) \leqslant d(F_2, G_2)$, if $\kappa_2/\kappa_1 \in \mathbf{K}_S$.

A metric must satisfy the following conditions:

- $d(F,G) = d(H_F(\cdot;\varphi), H_G(\cdot;\varphi)), \quad$ for all $\quad \varphi \in \mathbf{\Phi}$ and
- $d(F_1, G_1) \leqslant d(F_2, G_2)$, if $\kappa_2/\kappa_1 \in \mathbf{K}_S$.

The distance generated by the dispersive order

$$d_0(F,G) = \sup_{u\in(0,1)} \left| \ln \frac{fF^{-1}(u)}{gG^{-1}(u)} \right| .$$

Theorem 4

If $\kappa_i(u) \geqslant 1$, $i = 1, 2$, and $\kappa_2(u)/\kappa_1(u) \geqslant 1$, then

$$d_0(F_1, G_1) \leqslant d_0(F_2, G_2).$$

Theorem 4

If $\kappa_i(u) \geqslant 1$, $i = 1, 2$, and $\kappa_2(u)/\kappa_1(u) \geqslant 1$, then

$$d_0(F_1, G_1) \leqslant d_0(F_2, G_2).$$

Theorem 5

If $\kappa_i(u)$, $i = 1, 2$, and $\kappa_2(u)/\kappa_1(u)$ are increasing, and also

$$\lim_{u \to 0^+} \kappa_i(u) = 1, \qquad i = 1, 2,$$

then

$$d_0(F_1, G_1) \leqslant d_0(F_2, G_2).$$

Example 1.

Let $\{F_\lambda(\cdot) = F(\cdot/\lambda), \lambda > 0\}$, where $F \in \mathcal{F}$ is fixed.
It is well known that this

$$\lambda_1 \leqslant \lambda_2 \quad \Rightarrow \quad F_{\lambda_1} \leqslant_{\mathrm{disp}} F_{\lambda_2}.$$

Since $f_\lambda F_\lambda^{-1}(u) = fF^{-1}(u)/\lambda$, then we have

$$d_0(F_{\lambda_1}, F_{\lambda_2}) = \ln \frac{\lambda_2}{\lambda_1}.$$

Example 2.

Let $\bar{F}(t) = \frac{1}{2}(e^{-2t} + e^{-t})$ and $\bar{G}(t) = e^{-t}$, $t > 0$. Then $F \leqslant_{lr} G$ and we have:

$$d_{hr}(F,G) = \sup_{t>0} \left| \ln \frac{2e^{-2t}}{e^{-2t} + e^{-t}} \right| = \sup_{t>0} \left| \ln \frac{2}{e^{t} + 1} \right| = \ln 2,$$

and

$$d_{lr}(F,G) = \sup_{t>0} \left| \ln \frac{e^{-t}}{e^{-2t} + \frac{1}{2}e^{-t}} \right| = \ln \left(\frac{2}{3} \right).$$

Example 2.

Since $F \leqslant_{hr} G$ and F and G are DFR, then $F \leqslant_{disp} G$.
Simple calculations show that $fF^{-1}(u) = (9 - 8u + \sqrt{9 - 8u})/4$, $u \in (0, 1)$, and hence

$$d_0(F, G) = \sup_{u \in (0,1)} \left| \ln \frac{9 - 8u + \sqrt{9 - 8u}}{4(1 - u)} \right| = \ln \left(\frac{2}{3} \right) = d_{lr}(F, G).$$

Example 3 distance for residual lifetime distribution

Consider the residual lifetime distributions F_{t_1}, F_{t_2} from *IFR* class. It is well known that $F_{t_2} \leqslant_{\text{disp}} F_{t_1}$, for all $0 < t_1 \leqslant t_2$. We have:

$$F_t^{-1}(x) = F^{-1}(1 - (1 - x)\bar{F}(t)) - t,$$

hence

$$f_t F_t^{-1}(x) = \frac{fF^{-1}(1 - (1 - x)\bar{F}(t))}{\bar{F}(t)},$$

thus

$$d_0(F_{t_1}, F_{t_2}) = \sup_x \left| \ln \left[\frac{\bar{F}(t_1)}{\bar{F}(t_2)} \cdot \frac{fF^{-1}(1 - (1 - x)\bar{F}(t_2))}{fF^{-1}(1 - (1 - x)\bar{F}(t_1))} \right] \right|.$$

Example 3 distance for residual lifetime distribution

- Let $\bar{F}(x) = \exp\{-x^\alpha\}$ Weibull survival function with shape parameter $\alpha > 1$, then F is IFR. We can compute

$$d_0(F_{t_1}, F_{t_2}) = \sup_x \left| \frac{\alpha}{\alpha - 1} \ln \left[\frac{-\ln(1-x) + t_2^\alpha}{-\ln(1-x) + t_1^\alpha} \right] \right| = \frac{\alpha^2}{\alpha - 1} \ln \frac{t_2}{t_1}.$$

- Let be given double exponential distribution with survival function $\bar{F}(x) = \exp\{-\alpha e^x\}$, $\alpha > 0$. Here we get:

$$d_0(F_{t_1}, F_{t_2}) = \sup_{x \in (0,1)} \left| \ln \left[\frac{-\ln(1-x) + \alpha e^{t_2}}{-\ln(1-x) + \alpha e^{t_2}} \right] \right| = t_2 - t_1.$$

References

- J. Bartoszewicz, M. Benduch, (2008), *Some properties of the generalized TTT transform*, J. Statist. Plann. Inference, **139**, 2008–2017.
- E.L. Lehmann i J. Rojo, (1992), *Invariant directional orderings*, Ann. Statist. **20**, 2100-2110.
- X. Li,M. Shaked, (2007), *A general family of univariate stochastic orders*, J. Statist. Plann. Inference **137**, 3601-3610.